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Artice M. Davis

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The field concept in Ampère's magnetostatics

Artice M. Davis[a)]
P. O. Box 828, Shady Cove, Oregon 97539

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We update Ampère's theory using vector notation and derive his expression for the force between two current elements. We assume that the two elements are in different current loops and integrate over one to obtain the force on a differential element in the second. This procedure allows us to define the magnetic field in a natural manner and to derive the Lorentz force for a current segment. We equate the magnetic moments of current and permanent magnet dipoles and show that Biot and Savart could have performed their experiment using a small current loop, thus establishing the Biot-Savart law as a consequence of Ampère's theory.
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I. INTRODUCTION

Ampère, in a series of clever experiments and conjectures[1–4] in the 1820s, derived an expression for the force between two current elements $\vec{m}=Id\vec{l}$ and $\vec{n}=I'd\vec{l}'$. In vector notation the force $\vec{F}_{mn}(\vec{r})$ on element $\vec{m}$ caused by element $\vec{n}$ is

$$\vec{F}_{mn}(\vec{r}) = K\left[\frac{2(\vec{m}\cdot\vec{n})}{r^3} - \frac{3(\vec{m}\cdot\vec{r})(\vec{n}\cdot\vec{r})}{r^5}\right]\vec{r}, \tag{1}$$

where $\vec{r}$ is the position of element $\vec{n}$ relative to element $\vec{m}$ and K is a constant to be determined by the choice of units. Equation (1) is rarely used these days for several reasons. Ampère's derivation was complex,[5] and commentators on his work have mostly adopted his original style with its prolixity of expression and coordinate-based mathematics. Those few who use vector analysis are interested in research questions. As a result there is no readily available clear exposition of Ampère's theory couched in modern terms.

An even more important shortcoming is Ampère's failure to incorporate the idea of a field.[6] Nineteenth century physicists were divided into two camps. Continental physicists adopted the "action at a distance" viewpoint, whereas English investigators such as Maxwell followed Faraday's lead by thinking of interactions as being mediated by a field that propagated through the ether. Ampère, being French, formulated his interaction equation as an action at a distance. When Michelson showed the nonexistence of the ether, it became clear that Faraday and Maxwell were wrong. The ether was discarded, but the idea of a field remained.

As a result, the pedagogical foundations of field theory were arrived at in a disorganized way. It is now standard practice in introductory texts that do not use relativity[7,8] to define the magnetic field $\vec{B}$ using the Lorentz force law

$$\vec{F}(\vec{r}) = \vec{v}\times\vec{B}_s(\vec{r}), \tag{2}$$

where $\vec{F}(\vec{r})$ is the force per unit charge on a charge moving with velocity $\vec{v}$ and $\vec{r}$ is the position of the charge relative to an arbitrary origin. $\vec{B}_s(\vec{r})$ is given by the Biot-Savart law

$$\vec{B}_s(\vec{r}) = K_s\oint_C \frac{I'd\vec{l}'\times\vec{R}}{R^3}, \tag{3}$$

where $\vec{R}=\vec{r}-\vec{r}'$ and $d\vec{l}'=d\vec{r}'$ is a differential length of a conductor at position $\vec{r}$ in the closed circuit C. This approach has several defects. Although the Lorentz force law can be derived for constant $\vec{v}$ from the Lorentz transformation of special relativity, it has no experimental foundation[9] within the classical theory—it is an *ansatz* that is conceptually difficult for a student to accept.

Another defect lies in the Biot-Savart law itself, which does not involve Eq. (2)[10] Biot and Savart used a magnetized needle to investigate the force field of a long straight conductor carrying a constant current. They then bent the conductor so that it formed an angle near the needle and repeated their experiment. On the basis of their results they hypothesized[11] a relation that can be cast into the form of Eq. (3) and a *different* Eq. (2), namely

$$\vec{F}_s(\vec{r}) = n_0\vec{B}_s(\vec{r}), \tag{4}$$

the force on a hypothetical magnetic pole of strength n_0. Biot and Savart actually only inferred Eq. (4) because the response variable in their experiment was the torque on their magnetic needle. As we will show, Eqs. (3) and (4) can be used to derive the torque as

$$\vec{T}_s(\vec{r}) = \vec{n}\times\vec{B}_s(\vec{r}), \tag{5}$$

where $\vec{n}=n_0\vec{d}$ is the dipole moment of the needle with pole strength n_0 and vector length $\vec{d}$. Thus, their $\vec{B}_s$ is a hybrid field which gives the force on a hypothetical magnetic pole of pole strength n_0 through Eq. (4) rather than via the Lorentz force of Eq. (2). The latter in its current element form $\vec{F}(\vec{r})=Id\vec{l}\times\vec{B}_s(\vec{r})$ and Eq. (3) actually represent in modern notation a result that Grassman[12] derived from Ampère's law in 1845 using somewhat dubious assumptions and mathematics.

One thing is clear: the provenance of the modern Biot-Savart approach to teaching magnetostatics is *far* from clear. In contrast, Ampère's equation (1) is phrased in terms of current elements and fits much better into our modern view of magnetism. In this paper we will demonstrate that from an appropriate perspective Ampère's theory has none of the shortcomings of the conventional approach. We will first present a careful derivation of Eq. (1) using vector terminology and the isotropy of physical space. In all other respects we will maintain Ampère's flow of logic. After deriving Eq. (1), we will apply it in integrated form to find the force on a given current element due to a complete source circuit. We show that this step leads naturally to the usual definition of the magnetic field in Eq. (3) and to a simple derivation of

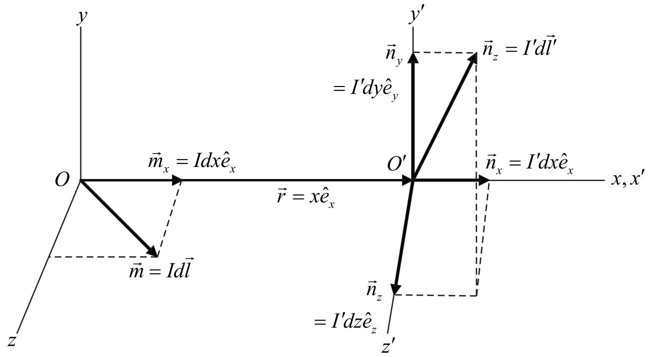

Fig. 1. Two arbitrarily situated current elements after the first application of isotropy.

the Lorentz force in Eq. (2). The result is a logically coherent approach to magnetostatics.[13]

We next apply our newly developed Ampérian field theory to calculate the torque on a small current loop and use it to show that Biot and Savart could well have used it in their experiment, whose outcome can thus be interpreted as a verification of Ampére's theory. We end the paper with a brief discussion of the present status of magnetostatic field theory.

II. THE AMPÈRE FORCE LAW FOR A PAIR OF CURRENT ELEMENTS

We will rely heavily upon the isotropy[14] and homogeneity of space which imply that our equations should be independent of our choices of direction for the three orthogonal coordinate axes and of the position of their origin. We can express the idea of isotropy as follows. Suppose we have derived a relation among m vector variables in the form

$$F(\vec{x}_1,\vec{x}_2, \ldots ,\vec{x}_m) = 0, \tag{6}$$

where F is either a vector or scalar relation. Let $Q_{\vec{v},\phi}$ be an operator that rotates any vector by an angle ϕ about the axis $\vec{v}$. If we can write

$$F(Q_{\vec{v},\phi}\vec{x}_1,Q_{\vec{v},\phi}\vec{x}_2, \ldots ,Q_{\vec{v},\phi}\vec{x}_m) = 0 \tag{7}$$

for any arbitrary choices for $\vec{v}$ and ϕ, then we say that the relation (6) is *isotropic*. Furthermore, if

$$F(\vec{x}_1 - \vec{x}_0,\vec{x}_2 - \vec{x}_0, \ldots ,\vec{x}_m - \vec{x}_0) = 0 \tag{8}$$

for arbitrary choices of the shift parameter $\vec{x}_0$, we say that F is *shift-invariant*. Equations (7) and (8), respectively, codify the ideas of isotropy and homogeneity of space. We note that rotating all vectors by the same amount about the same axis is equivalent to rotating the basis vectors in the opposite directions the same amount about the same axis. Thus, we can interpret isotropy as either a rotation of all the vectors involved or as a change from one orthonormal basis to another with the same handedness.[15]

Our first use of isotropy lies in the very formulation of the problem tackled by Ampère, which consists of determining the interaction force of two arbitrarily situated differential elements of current as shown in Fig. 1. We assume that the position O' of the element $\vec{n}=I'd\vec{l}'$ relative to the position O of the element $\vec{m}=Id\vec{l}$ is along their common x,x' axis and that $\vec{m}$ lies in the xy plane. (Note that O' is also the origin of a "primed coordinate" system based at the current element $\vec{n}$.) The problem is to find the force on $\vec{m}$ caused by $\vec{n}$.

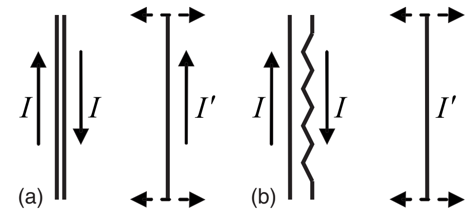

Fig. 2. The first two of Ampère's four experiments: (a) Experiment 1 and (b) Experiment 2.

Two experiments by Ampère. Ampère devised a sequence of four carefully conceived experiments[16] to provide clues to the nature of the force law. Figure 2 shows schematized sketches of the first two experiments. In the experiment shown in Fig. 2(a) he investigated the force that two closely spaced parallel current-carrying fixed conductors exert on a third parallel current-carrying conductor that is free to move. He observed no effect, whereas each current alone caused the mobile conductor to move. In the second experiment (shown in Fig. 2(b)) he replaced one of the fixed conductors with a zigzag conductor having small deviations from rectilinear. Again he observed no effect. From the first experiment he inferred that the force is reversed if the current in one of the conductors is reversed; from the second experiment he inferred that if both elements are decomposed into components, then the total force on one element is the superposition of all of the pairwise interaction forces acting individually. He also deduced from these experiments and a few reasonable assumptions that the force between two current elements is proportional to the product of their currents and also to the product of their lengths.

Consequences of Newtonian postulates. As he stated several times in his memoirs, Ampère was quite aware that it is impossible to confirm conjectures about differential current elements on the basis of experiments which must be made with complete circuits. He strongly believed, however, that the forces between them should obey Newton's third law. Therefore, he postulated that the total force on the system of two elements should be zero and that there should be no net torque that would cause a rotation. From these two assumptions he argued that the forces on the elements should be equal and opposite and should lie along the vector $\vec{r}$ in Fig. 1. This assumption has some major consequences, as we will see.

If we let $\vec{F}_{\alpha\beta}$ represent the vector force that component β of $\vec{n}$ exerts on component α of $\vec{m}$, then some of the $\vec{F}_{\alpha\beta}$ will be zero. Consider, for instance, the force $\vec{F}_{zx}$ shown in Fig. 3. Reversing the sign of I reverses the direction of both $\vec{m}_z$ and $\vec{F}_{zx}$. We apply our isotropy assumption by rotating the y, z

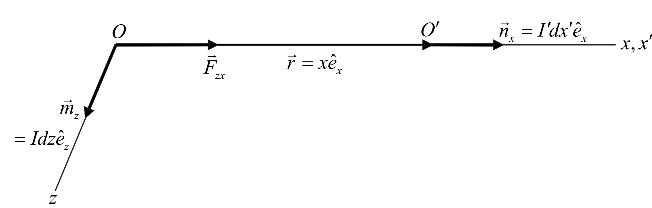

Fig. 3. Geometry of the $\vec{n}_x - \vec{m}_z$ interaction.

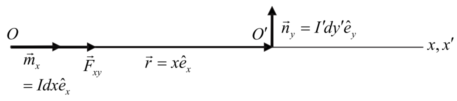

Fig. 4. Geometry of $\vec{n}_y - \vec{m}_z$.

plane by 180° around the x, x' axis. This rotation restores $\vec{m}_z$ to its original direction, while affecting neither $\vec{n}_x$ nor $-\vec{F}_{zx}$, which implies that $\vec{F}_{zx} = -\vec{F}_{zx}$, or $\vec{F}_{zx} = 0$.

Now let's apply the same type of reasoning to determine $\vec{F}_{xy}$, which is shown in Fig. 4. We reverse the sign of $\vec{I}'$ which reverses $\vec{n}_y$ and $\vec{F}_{xy}$, but has no effect on $\vec{m}_x$. We again rotate all vectors by 180° around the x, x' axis, thus restoring the original configuration of $\vec{m}_x$ and $\vec{n}_y$. Because these operations do not affect the value of $-\vec{F}_{xy}$, we must have $\vec{F}_{xy} = -\vec{F}_{xy}$ or $\vec{F}_{xy} = 0$. The same type of reasoning implies that $\vec{F}_{xz} = 0$.

There are two components of $\vec{m}$ and three components of $\vec{n}$ for a total of six possible interactions; three interactions are zero, leaving three that are nonzero: $\vec{F}_{xx}, \vec{F}_{zy}, \vec{F}_{zz}$. By invoking superposition of the effects of these nonzero components, we can write

$$\vec{F}_{mn} = \vec{F}_{xx} + \vec{F}_{zy} + \vec{F}_{zz} = [F_{xx} + F_{zy} + F_{zz}]\hat{e}_x. \tag{9}$$

According to Ampère we know a bit more: each force is proportional to the product of the currents and also to their differential lengths, that is, to the product of the constituents of $\vec{m}$ and $\vec{n}$. Because each force is a function of the separation x, we can write Eq. (9) as

$$\vec{F}_{mn} = [\alpha'(x)m_x n_x + \beta(x)m_z n_y + \gamma(x)m_z n_z]\hat{e}_x, \tag{10}$$

where α', β, γ are functions of x to be determined. The lack of an arrow on a term refers to the magnitude of the associated vector component. By adding and subtracting $\gamma(x)m_x n_x$ and adding the zero quantity $\gamma(x)m_y n_y$ (m_y is zero because of our choice of coordinate system), we have

$$\begin{aligned}\vec{F}_{mn} = [&\alpha(x)m_x n_x + \beta(x)m_z n_y \\ &+ \gamma(x)(m_x n_x + m_y n_y + m_z n_z)]\hat{e}_x,\end{aligned} \tag{11}$$

where $\alpha(x) = \alpha'(x) - \gamma(x)$. By direct calculation, we can easily verify the validity of the following form of Eq. (11):

$$\begin{aligned}\vec{F}_{mn} = [&\alpha(x)(\vec{m}\cdot\hat{e}_x)(\vec{n}\cdot\hat{e}_x) + \beta(x)(\vec{n}\times\vec{m}\cdot\hat{e}_x) \\ &+ \gamma(x)(\vec{m}\cdot\vec{n})]\hat{e}_x.\end{aligned} \tag{12}$$

More implications of isotropy. We will invoke isotropy again, but before we do, a slight digression is helpful. Thus far, we have referred everything to the orthonormal basis $B = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, where we have just substituted 1 for x, 2 for y, and 3 for z to better express sums in terms of indices. Suppose we now change to a new orthonormal basis $B' = \{e'_1, \hat{e}'_2, \hat{e}'_3\}$. We can express any vector $\hat{e}'_j$ in B' in terms of those in B by $\hat{e}'_j = q_{ij}\hat{e}_i$ using the summation convention on the indices (as we will continue to do). Writing $\hat{e}'_j \cdot \hat{e}'_k = (q_{ij}\hat{e}_i)\cdot(q_{lk}\hat{e}_l) = q_{ij}q_{lk}(\hat{e}_i \cdot \hat{e}_l)$ and using the orthonormality of both bases, we have $\delta_{jk} = q_{ij}q_{lk}\delta_{il} = q_{ij}q_{ik}$. We can express any vector in terms of its B components by $\vec{v} = v_i\hat{e}_i$ or in terms of its B' components by $\vec{v} = v'_j\hat{e}'_i$. This enables us to write $v_i\hat{e}_i = v'_j\hat{e}'_i = v'_j q_{ij}\hat{e}_i$, so the components are related by $v_i = q_{ij}v'_j$.

Let's see what happens to the scalar product when we change bases. Given any two vectors expressed in terms of the B basis by $\vec{u} = u_i\hat{e}_i$ and $\vec{v} = v_j\hat{e}_j$, we define their scalar product by $\vec{u}\cdot\vec{v} = u_i v_i$. We rewrite it in terms of the B' basis as $\vec{u}\cdot\vec{v} = u_i v_i = q_{ij}q_{ik}u'_j v'_k = \delta_{jk}u'_j v'_k = u'_j v'_j$, the penultimate equality being justified by the orthonormality of B'. Thus, the inner product is invariant under a change from one orthonormal basis to another.

The first and third terms in Eq. (12) involve scalar products. The second term includes a cross product. To investigate it under a change of orthonormal basis we again choose two vectors $\vec{u}$ and $\vec{v}$ as in our investigation of the scalar product. Relative to B their vector product is

$$\vec{u}\times\vec{v} = \hat{e}_i\varepsilon_{ijk}u_j v_k, \tag{13}$$

where ε_{ijk} is the Levi-Civita epsilon symbol, which is zero unless i, j, and k are all distinct. If they are distinct, then $\varepsilon_{ijk} = 1$ if i, j, k form an even permutation of 1, 2, 3 and $\varepsilon_{ijk} = -1$ if the permutation is odd. If we insist that B' form a right-handed system as is the case for B, then we must have $\hat{e}'_j \times \hat{e}'_k = \hat{e}'_i\varepsilon_{ijk}$. Note that $\hat{e}'_j \times \hat{e}'_k = (q_{mj}\hat{e}_m)\times(q_{nk}\hat{e}_n) = q_{mj}q_{nk}(\hat{e}_m \times \hat{e}_n) = q_{mj}q_{nk}(\hat{e}_i\varepsilon_{imn})$ because B is right-handed and orthonormal. Thus, we can write

$$\begin{aligned}\vec{u}\times\vec{v} &= \hat{e}_i\varepsilon_{imn}u_m v_n \\ &= \hat{e}_i\varepsilon_{imn}(q_{mj}u'_j)(q_{nk}v'_k) \\ &= (q_{mj}q_{nk}\hat{e}_i\varepsilon_{imn})u'_j v'_k \\ &= (\hat{e}'_j \times \hat{e}'_k)u'_j v'_k \\ &= \hat{e}'_i\varepsilon_{ijk}u'_j v'_k.\end{aligned} \tag{14}$$

Equation (14) shows that the cross product is invariant under an exchange of right-handed bases; this invariance, coupled with the invariance of the scalar product, says that the second term in Eq. (12) is also invariant. Hence, our interelement force $\vec{F}_{mn}$ in Eq. (12) is invariant under a change of right-handed orthonormal bases.

We can assume that x is positive in Fig. 1. If it were negative, we would rotate our coordinate system by 180° about the y axis, thus reversing the directions of both $\hat{e}_x$ and $\hat{e}_z$. This operation preserves the right-hand orientation of the basis, but changes the sign of x. Thus, we can consider the coefficients α, β, γ in Eq. (12) to be functions of $r = |\vec{r}| = x$. If we perform a general rotation of axes $\hat{e}_x$ transforms into $\hat{e}_r$, where $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$, and Eq. (12) assumes the form

$$\begin{aligned}\vec{F}_{mn} = [&\alpha(r)(\vec{m}\cdot\hat{e}_r)(\vec{n}\cdot\hat{e}_r) + \beta(x)(\vec{n}\times\vec{m}\cdot\hat{e}_r) \\ &+ \gamma(x)(\vec{m}\cdot\vec{n})]\hat{e}_r.\end{aligned} \tag{15}$$

Ampère's third experiment. To evaluate the coefficients $\alpha(r), \beta(r), \gamma(r)$ Ampère performed an experiment on three coaxial rings, which we will call his "three ring experiment" (see Fig. 5). Rings a and c are fixed, ring b is free to move, and all carry the same current. Let d be the diameter of ring a. Ampère positioned the center of ring b at a distance z to the right of a and gave it the diameter kd. He then adjusted the diameter of ring c to be k times that of b, that is $k^2 d$, and positioned its center a distance of kz to the right of b. Thus ring c received the same scaling relative to b that b was given relative to a. He discovered that ring b was in a posi-

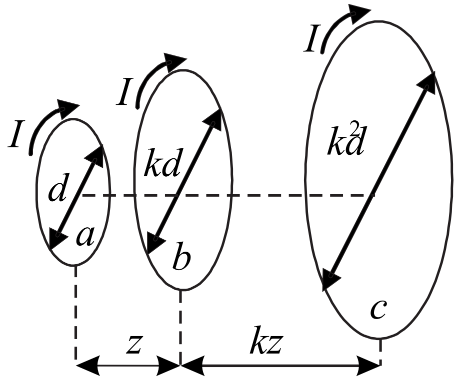

Fig. 5. Geometry of Ampère's three-ring experiment.

tion of stable equilibrium (if he moved it away from its initial location it returned there). From this experiment he inferred that a source conductor k times as far and k^2 times as long as another exerts the same force on a test conductor provided the source currents are the same. Ampère conjectured that the same scaling holds for *components of current elements*.

We can apply the results of Ampère's conclusions from this experiment, as did Ampère, by invoking superposition to find the coefficients one at a time. Thus, if we consider the first term in Eq. (15) and apply Ampère's conjecture, we can write

$$\alpha(r)(\vec{m}\cdot\hat{e}_r)(\vec{n}\cdot\hat{e}_r)=\alpha(kr)(\vec{m}\cdot\hat{e}_r)(k^2\vec{n}\cdot\hat{e}_r), \tag{16}$$

or $\alpha(r)=k^2\alpha(kr)$. (Think of the left-hand side as the force on ring b in Ampère's three ring experiment due to ring a and the right-hand side as the force on it due to ring c.) If we rewrite Eq. (16) in the form $\alpha(kr)=\alpha(r)k^{-2}$ and take the derivative of both sides with respect to k, we get $r\alpha'(kr)=-2\alpha(r)k^{-3}$. Let $k=1$. Then $r\alpha'(r)=-2\alpha(r)$, which we can solve to obtain $\alpha(r)=a/r^2$, where a is an arbitrary constant. The same analysis can be applied to each of the other terms in Eq. (15), thus transforming it into the form

$$\vec{F}_{mn}=\left[\frac{a(\vec{m}\cdot\hat{e}_r)(\vec{n}\cdot\hat{e}_r)}{r^2}+\frac{b(\vec{n}\times\vec{m}\cdot\hat{e}_r)}{r^2}+\frac{c(\vec{m}\cdot\vec{n})}{r^2}\right]\hat{e}_r, \tag{17}$$

where a, b, and c are constants yet to be determined. We will do so using Ampère's fourth experiment.

Ampère's fourth experiment. Ampère's fourth experiment, which we will call his "wire arc" experiment, is illustrated in the sketch of Fig. 6. The thin radial line segments represent fixed conductors insulated from each other, and the heavy curved line segment represents a short conductor bent into an arc which rests on the two fixed conductors and makes electrical contact with them. The heavy vertical line represents a nonconductive rod which pins the wire arc to a vertical post (represented by the heavy dot) around which it is free to pivot. The arc can therefore move tangentially, but cannot lift up from the fixed conductors. A current I is established in the loop consisting of the two fixed conductors and the wire arc. When Ampère brought a closed current loop of arbitrary geometry near the wire arc, the latter did not move. This outcome showed that the force on a differential element due to an arbitrary closed conductor is always at right angles to the element.

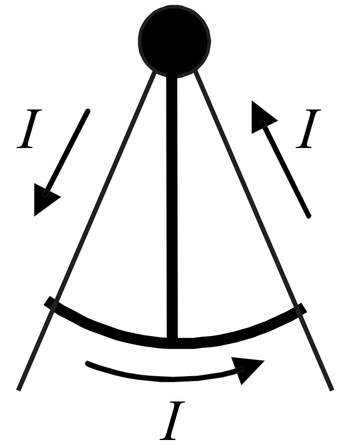

Fig. 6. Geometry of Ampère's wire arc experiment (top view).

We will use the result of the wire arc experiment to establish the values of two of the constants a, b, and c. We use the relation $\hat{e}_r=\vec{r}/r$ to rewrite Eq. (17) in the form

$$\vec{F}_{mn}=\left[\frac{a(\vec{m}\cdot\vec{r})(\vec{n}\cdot\vec{r})}{r^5}+\frac{b(\vec{n}\times\vec{m}\cdot\vec{r})}{r^4}+\frac{c(\vec{m}\cdot\vec{n})}{r^3}\right]\vec{r}. \tag{18}$$

Because $\vec{m}=Id\vec{l}$ and $\vec{n}=I'd\vec{l}'$, Eq. (18) becomes

$$\vec{F}_{mn}=II'\left[\frac{a(d\vec{l}\cdot\vec{r})(d\vec{l}'\cdot\vec{r})}{r^5}+\frac{b(d\vec{l}'\times d\vec{l}\cdot\vec{r})}{r^4}+\frac{c(d\vec{l}\cdot d\vec{l}')}{r^3}\right]\vec{r}. \tag{19}$$

Recall that we have used isotropy to go to a basis in which $\vec{r}$ has a general direction. We now imagine the current element $\vec{n}$ to be a component of a complete loop C as shown in Fig. 7 and change the notation slightly to $\vec{n}=I'd\vec{r}$. This notational change means that Eq. (19) can be written in the form

$$\vec{F}_{mn}=II'\left[\frac{a(d\vec{l}\cdot\vec{r})(d\vec{r}\cdot\vec{r})}{r^5}+\frac{b(d\vec{r}\times d\vec{l}\cdot\vec{r})}{r^4}+\frac{c(d\vec{l}\cdot d\vec{r})}{r^3}\right]\vec{r}. \tag{20}$$

For convenience we modify the second component by invoking the identity

$$d\vec{r}\times d\vec{l}\cdot\vec{r}=d\vec{r}\cdot d\vec{l}\times\vec{r} \tag{21}$$

to rewrite Eq. (20) as

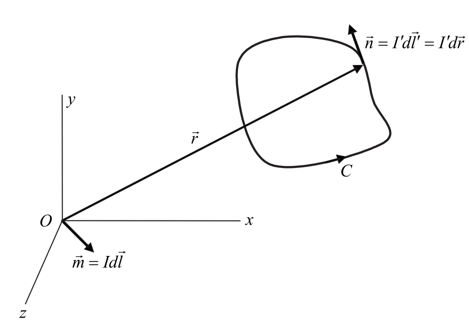

Fig. 7. The effect of a complete current loop on a differential current element in another loop.

$$\vec{F}_{mn} = II'\left[\frac{a(d\vec{l}\cdot\vec{r})(d\vec{r}\cdot\vec{r})}{r^5} + \frac{b(d\vec{r}\cdot d\vec{l}\times\vec{r})}{r^4} + \frac{c(d\vec{l}\cdot d\vec{r})}{r^3}\right]\vec{r}. \tag{22}$$

We now simplify the problem by invoking isotropy again, and assume that the x axis is along $\vec{m}$. Ampère's wire arc experiment says that if we integrate the x component of the force on $\vec{m}$ (the component of the force parallel to $\vec{m}$), then the result will be zero. The integral around the contour C of the x component of Eq. (22) is

$$\begin{aligned}\oint_C F_x &= II'\oint_C\left[\frac{ax(d\vec{l}\cdot\vec{r})(d\vec{r}\cdot\vec{r})}{r^5}\right.\\ &\quad\left.+\frac{bx(d\vec{r}\cdot d\vec{l}\times\vec{r})}{r^4} + \frac{cx(d\vec{l}\cdot d\vec{r})}{r^3}\right]\\ &= II'\oint_C\left[\frac{ax(d\vec{l}\cdot\vec{r})\vec{r}}{r^5}\right.\\ &\quad\left.+\frac{bx(d\vec{l}\times\vec{r})}{r^4} + \frac{cxd\vec{l}}{r^3}\right]\cdot d\vec{r}.\end{aligned} \tag{23}$$

Equation (23) has the form

$$\oint_C F_x = II'\oint_C \vec{Q}(\vec{r})\cdot d\vec{r}, \tag{24}$$

where

$$\vec{Q}(\vec{r}) = \frac{ax(d\vec{l}\cdot\vec{r})\vec{r}}{r^5} + \frac{bx(d\vec{l}\times\vec{r})}{r^4} + \frac{cxd\vec{l}}{r^3}. \tag{25}$$

In our new coordinate system we can write

$$d\vec{l} = dl\hat{e}_x, \tag{26a}$$

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z, \tag{26b}$$

$$d\vec{r} = dx\hat{e}_x + dy\hat{e}_y + dz\hat{e}_z. \tag{26c}$$

Thus,

$$d\vec{l}\cdot\vec{r} = xdl, \tag{27a}$$

$$d\vec{l}\times\vec{r} = dl\hat{e}_x\times(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z) = dl(y\hat{e}_z - z\hat{e}_y). \tag{27b}$$

Equation (25) then assumes the form

$$\begin{aligned}\vec{Q}(\vec{r}) &= \frac{ax^2(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z)}{r^5}dl\\ &\quad+\frac{bx(y\hat{e}_z - z\hat{e}_y)}{r^4}dl + \frac{cx\hat{e}_x}{r^3}dl\\ &= \left[\frac{ax^3}{r^5} + \frac{cx}{r^3}\right]dl\hat{e}_x + \left[\frac{ax^2y}{r^5} - \frac{bxz}{r^4}\right]dl\hat{e}_y\\ &\quad+\left[\frac{ax^2z}{r^5} + \frac{bxy}{r^4}\right]dl\hat{e}_z\\ &= Q_x(\vec{r})\hat{e}_x + Q_y(\vec{r})\hat{e}_y + Q_z(\vec{r})\hat{e}_z.\end{aligned} \tag{28}$$

The wire arc experiment implies that the integral in Eq. (23) is zero for any contour C. We can therefore apply Stokes's theorem to assert that $\nabla\times\vec{Q}(\vec{r})=0$, which in rectangular coordinates has the form

$$\begin{aligned}\nabla\times\vec{Q}(\vec{r}) &= (\partial_y Q_z - \partial_z Q_y)\hat{e}_x - (\partial_x Q_z - \partial_z Q_x)\hat{e}_y\\ &\quad+(\partial_x Q_y - \partial_y Q_x)\hat{e}_z = 0.\end{aligned} \tag{29}$$

Equation (29) implies that the individual components are zero. The x component of the curl is

$$\begin{aligned}(\nabla\times\vec{Q})_x &= \partial_y Q_z - \partial_z Q_y\\ &= bxdl\left[\frac{2}{r^4} - \frac{4(y^2+z^2)}{r^6}\right],\end{aligned} \tag{30}$$

which implies that $b=0$. We use this value of b to calculate the y component and obtain

$$(\nabla\times\vec{Q})_y = \partial_z Q_x - \partial_x Q_z = -(2a+3c)xzdl\frac{1}{r^5}. \tag{31}$$

The only way the right-hand side of Eq. (31) can be zero is for the coefficient to be zero. Hence $c=2K$ and $a=-3K$, where K is an arbitrary constant. With these values, the equation $(\nabla\times\vec{Q})_z=0$ is satisfied identically. We use these results in Eq. (22) and obtain

$$\vec{F}_{mn} = KII'\left[\frac{2(d\vec{l}\cdot d\vec{l}')}{r^3} - \frac{3(d\vec{l}\cdot\vec{r})(d\vec{l}'\cdot\vec{r})}{r^5}\right]\vec{r}. \tag{32}$$

Equation (32) is the classical Ampère expression for the force between two differential segments of current-carrying conductor.

III. THE MAGNETIC FIELD AND THE LORENTZ FORCE

We now pause briefly for a mathematical aside and define[17]

$$\vec{f}(\vec{r}) = \frac{(\vec{a}\cdot\vec{r})\vec{r}}{r^3}, \tag{33}$$

and use the properties of differentials to write

$$d\vec{f} = \frac{(\vec{a}\cdot d\vec{r})\vec{r}}{r^3} + \frac{(\vec{a}\cdot\vec{r})d\vec{r}}{r^3} - \frac{3(\vec{a}\cdot\vec{r})\vec{r}dr}{r^4}. \tag{34}$$

We know that $d(r^2/2)=rdr=\vec{r}\cdot d\vec{r}$, so by multiplying and dividing the last term of Eq. (34) by r, we can write

$$d\vec{f} = \frac{(\vec{a}\cdot d\vec{r})\vec{r}}{r^3} + \frac{(\vec{a}\cdot\vec{r})d\vec{r}}{r^3} - \frac{3(\vec{a}\cdot\vec{r})(\vec{r}\cdot d\vec{r})\vec{r}}{r^5}. \tag{35}$$

The last term looks like one of the terms in the Ampère force, and becomes more so when we let $\vec{a}=d\vec{l}$ and recognize that $d\vec{r}=d\vec{l}'$. If we make these substitutions and solve for the last term in Eq. (35), we obtain

$$\frac{3(d\vec{l}\cdot\vec{r})(d\vec{l}'\cdot\vec{r})\vec{r}}{r^5} = \frac{(d\vec{l}\cdot d\vec{l}')\vec{r}}{r^3} + \frac{(d\vec{l}\cdot\vec{r})d\vec{l}'}{r^3} - d\vec{f}(\vec{r}). \tag{36}$$

Substitution of this expression in Eq. (32) results in

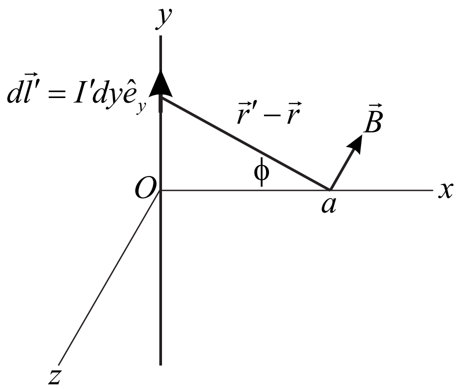

Fig. 8. The magnetostatic field of an infinitely long straight conductor.

$$\vec{F}_{mn} = KII'\left[\frac{(d\vec{l}\cdot d\vec{l}')\vec{r} - (d\vec{l}\cdot\vec{r})d\vec{l}'}{r^3} + d\vec{f}(\vec{r})\right]. \tag{37}$$

Recalling the identity $d\vec{l}\times(d\vec{l}'\times\vec{r}) = (d\vec{l}\cdot\vec{r})d\vec{l}' - (d\vec{l}\cdot d\vec{l}')\vec{r}$ and letting $\vec{R} = -\vec{r}$, we find that

$$\vec{F}_{mn} = KII'\left[\frac{d\vec{l}\times(d\vec{l}'\times\vec{R})}{R^3} + d\vec{f}(\vec{R})\right]. \tag{38}$$

The total force $\vec{F}_m$ on the differential segment $\vec{m} = Id\vec{l}$ at the origin can be obtained by integrating around the closed circuit C on which $\vec{n} = I'd\vec{l}'$ is assumed to constitute an element (see Fig. 7). The result is

$$\vec{F}_m = Id\vec{l}\times\oint_C K\frac{d\vec{l}'\times\vec{R}}{R^3}, \tag{39}$$

where the exact differential $d\vec{f}(\vec{R})$ has integrated to zero. The final step is to invoke the homogeneity of space, translate the element $\vec{m}$ to the general position $\vec{r}$, and let $\vec{r}'$ be the vector position of the source element $\vec{n}$ relative to the origin. [We have redefined the symbol $\vec{r}$ to be consistent with the literature with the result that $\vec{R} = \vec{r} - \vec{r}'$ in Eq. (39).] The latter becomes

$$\vec{F}_m(\vec{r}) = Id\vec{l}\times\left[K\oint_C\frac{d\vec{l}'\times\vec{R}}{R^3}\right] = Id\vec{l}\times\left[K\oint_C\frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\right], \tag{40}$$

which is the same form as Eq. (39) but with the new interpretation for $\vec{R}$. If we define the magnetic field vector by

$$\vec{B}(\vec{r}) = K\oint_C\frac{d\vec{l}'\times\vec{R}}{R^3} = K\oint_C\frac{d\vec{l}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}, \tag{41}$$

we can rewrite Eq. (40) in terms of the Lorentz force on a current-carrying conductor of length $d\vec{l}$:

$$\vec{F}_m(\vec{r}) = Id\vec{l}\times\vec{B}(\vec{r}). \tag{42}$$

To evaluate the constant K we find the magnetic field a perpendicular distance a from an infinitely long straight conductor (see Fig. 8). As shown in introductory texts, we can use Eq. (41) to derive

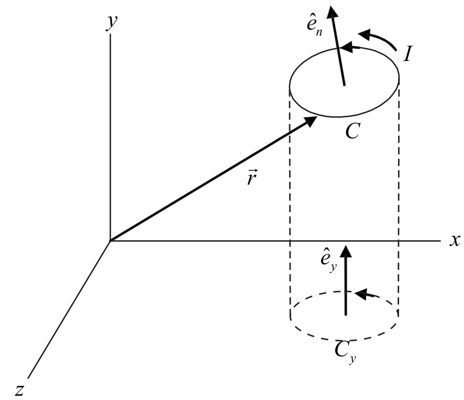

Fig. 9. Relative orientations of a loop and its projection in a coordinate plane.

$$\vec{B} = \frac{2KI'}{a}(-\hat{e}_z). \tag{43}$$

The force on a current element at a oriented parallel to the long straight conductor and carrying current I in the same direction is given by

$$\vec{F}_m = Idy\hat{e}_y\times\vec{B} = \frac{2KII'}{a}dy(-\hat{e}_x). \tag{44}$$

Thus, there is an attractive force of magnitude $2KII'/a$ per unit length.

In SI units this result is used to define the Ampère of current as follows: if $I = I' = 1$ A, then the force per unit length on a current element 2 m away from the infinite length conductor is 10^{-7} N/m. This definition is equivalent to requiring that $K = 10^{-7}$ N/A^2. A factor of 4π is usually included to simplify other more complicated, equations. We therefore define $\mu_0 = 4\pi K$ and rewrite Eq. (41) as

$$\vec{B}(\vec{r}) = \oint_C\frac{\mu_0 I'd\vec{l}'\times(\vec{r}-\vec{r}')}{4\pi|\vec{r}-\vec{r}'|^3} = \frac{\mu_0}{4\pi}\oint_C\frac{I'd\vec{l}'\times\vec{R}}{R^3}, \tag{45}$$

where $\mu_0 = 4\pi\times10^{-7}$ N/A^2. We have now completed a major part of the task we set out to do: we have defined the magnetostatic field and derived the Lorentz force law for a current element completely within Ampère's theory.

IV. DIPOLES, MAGNETIC MOMENT, AND THE BIOT-SAVART LAW

Consider a small planar current loop C situated in a magnetic field. Assume that the small surface defined by C has a unit normal $\hat{e}_n$ consistent with the right-hand rule and the orientation of C. The loop geometry is shown in Fig. 9.[18] The torque on a differential segment of this loop is

$$d\vec{T} = \vec{r}\times\vec{F}_m = \vec{r}\times[Id\vec{r}\times\vec{B}(\vec{r})] = Id\vec{r}[\vec{r}\cdot\vec{B}(\vec{r})] - I\vec{B}(\vec{r})[\vec{r}\cdot d\vec{r}]. \tag{46}$$

Because $\vec{r}\cdot d\vec{r} = d[r^2/2]$, it will integrate to zero around a closed loop. Thus, the total torque on the loop is given by

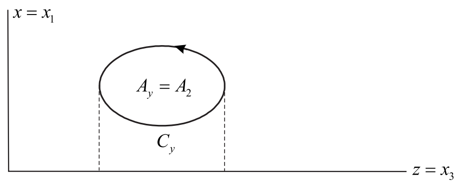

Fig. 10. The projection of C onto the z-x plane.

$$\vec{T} = \oint_C \vec{r} \times \vec{F}_m = I \oint_C \vec{r} \cdot \vec{B}(\vec{r}) d\vec{r}. \tag{47}$$

We will assume that the loop is so tiny that $\vec{B}$ is constant over its extent, which gives

$$\vec{T} = IB_i \oint_C x_i d\vec{r} = IB_i \left[\oint_C x_i \, dx_j \right] \hat{e}_j, \tag{48}$$

where $\vec{B}=B_i\hat{e}_i$ and $x=x_1$, $y=x_2$, and $z=x_3$. Next, consider the integral $I_{ij}=\oint_C x_i\, dx_j$. The particular case in which $x_i=x_1=x$ and $x_j=x_3=z$ is illustrated in Fig. 9, and the projection curve is shown in more detail in Fig. 10. Note that if $\hat{e}_n \cdot \hat{e}_y > 0$, then C and C_y are both traversed in the positive direction relative to the unit normals; if $\hat{e}_n \cdot \hat{e}_y < 0$, they are traversed in opposite directions. For the case shown we have

$$\oint_C x_i \, dx_j = \oint_{C_y} x \, dz = -A_y, \tag{49}$$

where the signed area A_y is positive if $\hat{e}_n \cdot \hat{e}_y > 0$ and negative if $\hat{e}_n \cdot \hat{e}_y < 0$. In general, we can write

$$\oint_C x_i dx_j = \varepsilon_{ijk} A_k, \tag{50}$$

where the signed area A_k is positive if $\hat{e}_n \cdot \hat{e}_k > 0$ and negative if $\hat{e}_n \cdot \hat{e}_k < 0$. (Note that $x_1=x$, $x_3=z$, and $A_y=A_2$ in Eq. (48) so that the right-hand side is $\varepsilon_{132}A_2=-A_y$.) Thus, Eq. (48) becomes

$$\vec{T} = IB_i \varepsilon_{ijk} A_k \hat{e}_j = I\hat{e}_j \varepsilon_{jki} A_k B_i = I\vec{A} \times \vec{B}, \tag{51}$$

where $\vec{A}=A_x\hat{e}_x+A_y\hat{e}_y+A_z\hat{e}_z$. If we define the magnetic moment by

$$\vec{\mu} = I\vec{A}, \tag{52}$$

we have

$$\vec{T} = \vec{\mu} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{\vec{\mu} \times [I' d\vec{l}' \times \vec{R}]}{R^3}. \tag{53}$$

We think of an elementary current dipole as being a small current loop in which the area approaches zero while the current approaches infinity in such a way that $\vec{\mu}=I\vec{A}$ remains constant. Then Eq. (53) gives the torque on an elementary dipole.

The Biot-Savart experiment. We next consider the Biot-Savart experiment in which the field was measured by the torque on a small permanent magnet. We will place ourselves back in time, and look over the shoulders of Biot and Savart as they perform their famous experiment. Because their sensor is not a current element but a permanent magnet, we allow for the fact that their magnetic field might be different from the one considered by Ampère. We will, along with Biot and Savart, define the magnetic field in terms of the force on a hypothetical magnetic pole. Thus, we define the magnetic field vector $\vec{B}_S$ by

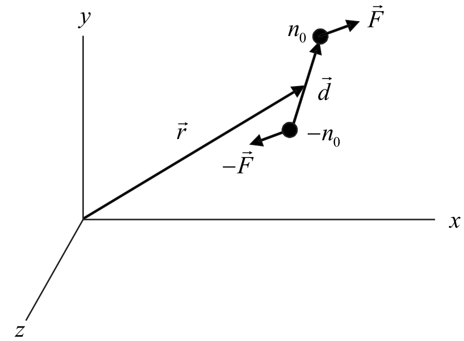

Fig. 11. Torque on a permanent magnetic dipole in a magnetic field.

$$\vec{F} = n_0 \vec{B}_S, \tag{54}$$

where n_0 is a constant—the "pole strength."

Consider a small permanent magnet consisting of two poles of opposite strengths $\pm n_0$ separated by the small vector distance $\vec{d}$ is in a magnetic field as shown in Fig. 11. We assume that $d=|\vec{d}|$ is so small that $\vec{B}_S$ is constant over the extent of the dipole. Then we can write the torque on this small dipole as

$$\vec{T} = (\vec{r} + \vec{d}/2) \times \vec{F} - (\vec{r} - \vec{d}/2) \times \vec{F} = \vec{d} \times n_0 \vec{B}' = n_0 \vec{d} \times B'. \tag{55}$$

We now define the magnetic moment of the small magnetic dipole to be $\vec{\mu}'=n_0\vec{d}$ and think of allowing the pole strength to approach infinity while the separation approaches zero.

We will now show that the Biot-Savart law is consistent with the theory expounded by Ampère. Biot and Savart hypothesized Eq. (3) on the basis of their measurements, but did not evaluate the constant K_S. If we combine Eq. (3) with Eq. (5), we have

$$\vec{T}(\vec{r}) = K_S \oint_C \frac{\vec{n} \times (I' d\vec{l}' \times \vec{R})}{R^3}. \tag{56}$$

We next infringe upon history a bit and insist that Biot and Savart repeat their experiment using Ampere's current loop. We insist, too, that they adjust its current in order for its magnetic moment to be the same as that of their magnetic needle. In symbolic terms we require that

$$\vec{n} = n_0 \vec{d} = I\vec{A} = \vec{\mu}. \tag{57}$$

We let the directions be the same and solve for n_0:

$$n_0 = IA/d. \tag{58}$$

Equation (58) is equivalent to choosing the unit of n_0 to be the Ampère-meter. If we select the undetermined constant K_S to have the value

$$K_S = \frac{\mu_0}{4\pi}, \tag{59}$$

then Eqs. (53) and (56) are identical. In other words, Biot and Savart could have performed their experiment with a small current loop and the outcome would have been unchanged. Hence, the Biot-Savart law is included in that of Ampère and affords experimental verification of the latter.

V. SUMMARY

We have developed an approach to magnetostatics based entirely on Ampère's theory and presented it in modern vector notation. This approach is consistent with the material currently being taught in introductory courses, but has an important advantage over the latter: unlike the Biot-Savart approach, the Ampérian field theory includes a derivation of the Lorentz force.

Although Ampère coined the term "electrodynamics," the field that he, Biot, and Savart investigated was the special case currently termed "magnetostatics." Thus, all of the experiments of that time—and almost all of those performed since then—are for constant (or slowly varying[19]) currents in which the source circuit is closed. Although Coulomb's law has been verified to extreme precision, the Biot-Savart-Ampère-Grassman law has not received the same attention.[20] The Lorentz force law, which is commonly used in the context of general time-varying fields, has apparently never been verified under such general conditions.[21] Finally, we observe (as have many others), that inferences about element pair forces from macroscopic experiments made on complete circuits are merely that: inferences, not certitudes. In a mathematical context[22] suppose a proposal is made that an element pair force is given by $\vec{F}_{mn}(\vec{r},d\vec{l},d\vec{r})$, where $\vec{m}=Id\vec{l}$ and $\vec{n}=I'd\vec{r}$. Let $d\vec{g}(\vec{r})$ be any exact differential. Then, as long as we restrict ourselves to an appropriate region of space (say one that is simply connected), we will have

$$\vec{F}_m=\oint_C \vec{F}_{mn}(\vec{r},d\vec{l},d\vec{r})=\oint_C [\vec{F}_{mn}(\vec{r},d\vec{l},d\vec{r})+dg(\vec{r})]. \qquad (60)$$

Any other element pair force that differs from the given one by an exact differential will result in the same single element force exerted by a complete circuit. That is, the element pair force is not uniquely determined from macroscopic experiments.

ACKNOWLEDGMENTS

A few years ago the author had the occasion to teach the introductory course in electromagnetics again after a lengthy hiatus and was taken aback when it was realized that students had problems understanding the most basic concepts—much more than had been the case several decades earlier. These problems were not entirely due to lack of analytical preparation, but arose in large measure from their need to see a more methodical construction of the theory from fundamental building blocks. The author therefore began to investigate the foundations of electrodynamics pedagogy more carefully, and the results led to an earlier publication[23] and the present one. The author would like to acknowledge the insight received early on in this quest from David Griffiths in personal communication, as well as from the intuitive insights gained from his text.[24] More recently, the author benefited a great deal from personal communications with Vladimir Onoochin, Alexander Kholmetsky, and David Dameron. The author also would like to express gratitude to two unknown reviewers whose suggestions contributed to the quality of this paper.

[a]Professor Emeritus, *San Jose State University*. Electronic mail: adavis@email.sjsu.edu

[1]Ampère described this work in several mèmoires in the 1820s. The one most often referenced is "Théorie mathématique des phénomènes electro-dynamique uniquement déduite de l'expérience," in Mémoires de l'Académie Royale des Sciences de la Institut de France, Année 1823, Tome VI, Paris, Chez Fermin Didot, Pére et Fils, 1827. A copy of this work and others by Ampère are available in French at ⟨www.ampere.cnrs.fr⟩. Portions of Ampère's work are reprinted in English in Ref. 2.

[2]R. A. R. Tricker, *Early Electrodynamics: The First Law of Circulation* (Pergamon, Oxford, 1965).

[3]L. Pearce Williams, "What were Ampère's earliest discoveries in electrodynamics?," Isis **74**, 492–508 (1983).

[4]C. Blondel, *A.-M. Ampere et la creation de l'electrodynamique (1820–1827)* (Bibliothèque Nationale, Paris, 1982).

[5]H. Erlichson, "André-Marie Ampère, the 'Newton of Electricity,' and how the simplicity criterion resulted in the disuse of his formula," Physis (Florence) **37**(1), 53–71 (2000).

[6]H. Erlichson, "The experiments of Biot and Savart concerning the force exerted by a current on a magnetic needle," Am. J. Phys. **66**, 385–391 (1998).

[7]J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, 4th ed. (Addison-Wesley, Reading, MA, 1993), p. 191.

[8]J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999), pp. 175–178.

[9]H. A. Lorentz, *The Theory of Electrons* (Dover, Mineola, 2003), pp. 14–15. After presenting the equation, Lorentz says "Like our former equations it is got by generalizing the results of electromagnetic experiments." Unfortunately, he does not say *which* experiments.

[10]P. Graneau and N. Graneau, *Newtonian Electrodynamics* (World Scientific, Singapore, 1996), pp. 28–34.

[11]See footnote 14 in Ref. 6, which quotes the Blunn translation of Ampère's memoir of Ref. 1 which, in turn, appears in Ref. 2. The item in question is an extensive footnote in Ampère's paper referring to the Biot-Savart experiment and to an important logical error that they committed. Blunn tones down Ampère's indictment of Biot and Savart considerably. In his original footnote in Ref. 1, Ampère makes it plain that Biot and Savart are guessing and complains about their failure to acknowledge a presentation by Felix Savary correcting the error.

[12]H. Grassman, "A new theory of electrodynamics," Ann. Phys. Chem. **64**(1), 1–18 (1845). An English translation appears in Ref. 2.

[13]This program was carried out in 1929 by M. Mason and W. Weaver in *The Electromagnetic Field* (Dover, New York, 1929), pp. 176–184. Their vector notation is dated, and they did not use isotropy arguments to support their manipulations. Thus, this train of thought is still relatively inaccessible by today's students.

[14]V. Gorini and A. Zecca, "Isotropy of space," J. Math. Phys. **11**(7), 2226–2230 (1970).

[15]A function is a special kind of relation. Thus, we can express $\vec{y}=F(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_m)$ in the form $\vec{y}-F(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_m)=0$, thus giving rise to a relation in $m+1$ variables. If $F(Q_{\vec{v},\phi}\vec{x}_1,Q_{\vec{v},\phi}\vec{x}_2,\ldots,Q_{\vec{v},\phi}\vec{x}_m)=Q_{\vec{v},\phi}\vec{y}$ for any choice of $\vec{v}$ and ϕ, we say F is an *isotropy* or is an *isotropic mapping*. If F is a scalar function, then no rotation operator multiplies y and F is a *scalar invariant*. We will use the definition of isotropy in this form applied to the element pair force function $\vec{F}_{mn}$.

[16]J. R. Hoffmann, "Ampère's invention of equilibrium apparatus: A response to experimental anomaly," Isis **20**, 309–341 (1987). References 1, 2, and 5 in the present paper (the last two in English) also describe Ampère's four experiments.

[17]The idea we develop here was inspired by C. Christodoulides, "Comparison of the Ampère and Biot-Savart magnetostatic force laws in their line-current-element forms," Am. J. Phys. **56**, 357–362 (1988).

[18]Ampere believed that magnetism was caused by circulating currents and did a number of experiments to demonstrate this idea. Several, including the effects of magnetism on a simple loop, are discussed in detail in A.-M. Ampère and J. Babinet, *Exposé des nouvelles découvertes sur l'electricité at le magnétisme* (Chez Méquignon-Marvis, Paris, 1822), pp. 6–15. It is available at ⟨books.google.fr⟩.

[19]J. Larsson, "Electromagnetics from a quasistatic perspective," Am. J. Phys. **75**, 230–239 (2007).

[20] G. Ruppeiner, M. Grossman, and A. Tafti, “Test of the Biot-Savart law to distances of 15 m,” Am. J. Phys. **64**, 698–705 (1996).
[21] Y.-S. Huang, “Has the Lorentz-covariant electromagnetic force law been directly tested experimentally?,” Found. Phys. Lett. **6**, 257–274 (1993).
[22] This observation and a specific application of our brief closing commentary was apparently first applied to prove the equivalence of the Biot-Savart and Ampère force laws by R. C. Lyness, “The equivalence of Ampère’s electrodynamic law and that of Biot and Savart,” Contemp. Phys. **4**, 453–455 (1963). Many others have since appeared, and an extensive list of references is given in Ref. 10, Chap. 2.
[23] A. M. Davis, “A generalized Helmholtz theorem for time-varying vector fields,” Am. J. Phys. **74**, 72–76 (2006).
[24] D. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, Upper Saddle River, 1999).

Lodestone. A Lodestone is a naturally occurring piece of magnetic iron oxide. It is often bound in a frame, and is oriented to place the magnetic poles at the ends. The word *magnet* comes from the region called Magnesia in Asia Minor. The word *lodestone* comes from the use of pieces of ore from Norway and Sweden that were suspended and used as guiding or leading stones; the Saxon word Læden means “to lead”. This example is in the collection of Westminster College in western Pennsylvania. The magnetic field is developed in the gap between the two iron vanes on the right-hand side. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)