Clase 3 - Martes 30/03/2021

La clase pasada vimos:

- Densidad de estados en general y para electrones libres
- Algunas propiedades termodinámicas en modelo de Sommerfeld
- Repaso del modelo de Drude

En esta clase vemos:

Efecto Hall en el modelo de Drude.

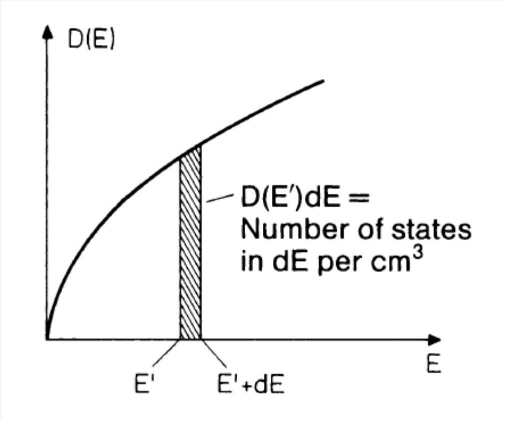
Conductividad AC (alterna u óptica) en el modelo de Drude.

Función dieléctrica: condición de transparencia de metales

Oscilaciones y frecuencia de plasma

REPASO

Teoría de Sommerfeld de los metales



Para el caso 3D

$$g(\mathcal{E}) = \frac{m}{\hbar^2 \pi^2} \sqrt{\frac{2m\mathcal{E}}{\hbar^2}}, \quad \mathcal{E} > 0;$$
$$= 0, \quad \mathcal{E} < 0.$$

$$g(\varepsilon) = \frac{2}{(2\pi)^3} \int_{\varepsilon = const.} \frac{d^2 S_\varepsilon}{|\mathrm{grad}_{\mathbf{k}} \varepsilon|}$$

Fórmula general

Modelo de Drude

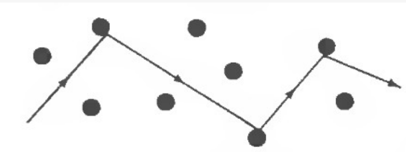


Figure 1.2 Trajectory of a conduction electron scattering off the ions, according to the naive picture of Drude.

$$\mathbf{j} = \sigma \mathbf{E}; \quad \sigma = \frac{ne^2\tau}{m}.$$

Modelo de Drude (1900)

Paul Drude
(1863-1906)

El origen de τ y su cálculo preciso pueden ser problemáticos.

Dos fenómenos que no dependen de τ

- Efecto Hall
- Conductividad AC: campo eléctrico/voltaje oscilante

Modelo de Drude

Se puede ver que el momento *promedio* de un electrón varía según:

$$\frac{d\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t).$$

Donde $\mathbf{f}(t)$ es la fuerza de Lorentz de los campos **E** y **B** aplicados.

El tiempo τ actúa como un tiempo de relajación del momento, suprimiendo la corriente que es proporcional a $\mathbf{p}(t)$:

$$\mathbf{j} = -\frac{ne\mathbf{p}(t)}{m}.$$

Modelo de Drude

Edwin Hall
(1855-1938)

El efecto Hall (1879):

Magneto-resistencia (transversa porque **H** es perp. al plano)

$$\rho(H) = \frac{E_x}{j_x}.$$

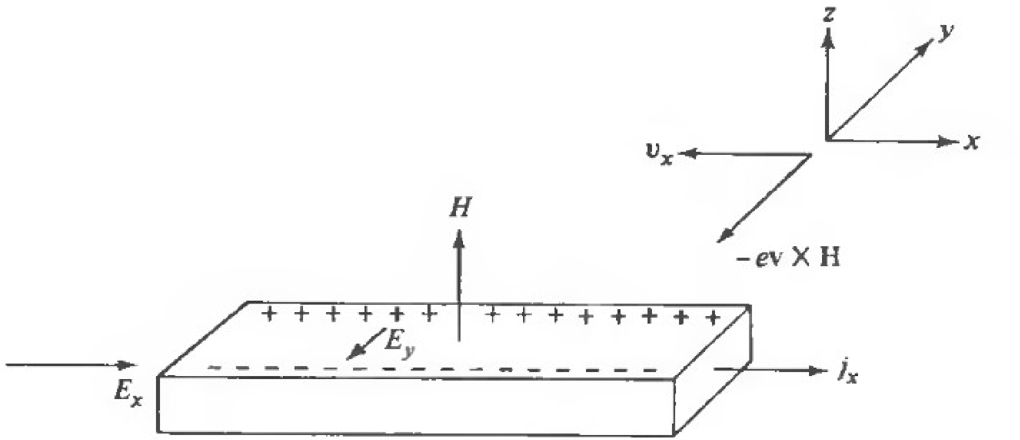


Figure 1.3
Schematic view of Hall's experiment.

$$-\frac{e}{c}\mathbf{v} \times \mathbf{H}$$

El campo transverso E_y aparece por la acumulación de cargas (que Hall no sabía qué eran)

$$R_H = \frac{E_y}{j_x H}.$$

Coeficiente de Hall

(Su signo depende del signo de la carga de los portadores)

→ Con el modelo de Drude se puede calcular el coeficiente de Hall, se obtiene (ejercicio 4):

$$R_H = -\frac{1}{nec}.$$

Modelos de Drude y Sommerfeld

Interludio: Notar las distintas **velocidades** de los electrones en un metal:

Velocidad térmica media: $\frac{1}{2}mv_0{}^2 = \frac{3}{2}k_B T.$ Depende fuertemente de la temperatura

of order 10^7 cm/sec. a T ambiente

Velocidad de Fermi:

ELEMENT	r_s/a_0	$\mathcal{E}_F$	T_F	k_F	v_F
Li	3.25	4.74 eV	5.51×10^4 K	1.12×10^8 cm^{-1}	1.29×10^8 cm/sec
Na	3.93	3.24	3.77	0.92	1.07

(about 1 percent of the velocity of light). Alta!

Velocidad de drift:

even in a current as large as

1 amp/mm^2, $v = j/ne$ is only of order 0.1 cm/sec.

Conductividad eléctrica AC de un metal

Supongamos que se aplica un campo eléctrico alterno u oscilante:

$$\mathbf{E}(t) = \mathrm{Re}\,(\mathbf{E}(\omega)e^{-i\omega t}).$$

Queremos calcular la corriente: $\mathbf{j} = -ne\mathbf{p}/m,$

Usamos la ecuación para la derivada del momento medio **p** : $\frac{d\mathbf{p}}{dt} = -\frac{\mathbf{p}}{\tau} - e\mathbf{E}.$

Y buscamos soluciones oscilatorias
con la misma frecuencia:

$$\mathbf{p}(t) = \mathrm{Re}\,(\mathbf{p}(\omega)e^{-i\omega t}).$$

$$\mathbf{j}(t) = \mathrm{Re}\,(\mathbf{j}(\omega)e^{-i\omega t}),$$

$$\mathbf{j}(\omega) = -\frac{ne\mathbf{p}(\omega)}{m}$$

Modelo de Drude

Conductividad eléctrica AC de un metal

$$\mathbf{E}(t) = \mathrm{Re}\,(\mathbf{E}(\omega)e^{-i\omega t}).$$

$$\mathbf{p}(t) = \mathrm{Re}\,(\mathbf{p}(\omega)e^{-i\omega t}).$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\mathbf{p}}{\tau} - e\mathbf{E}.$$

$$\longrightarrow \quad -i\omega\mathbf{p}(\omega) = -\frac{\mathbf{p}(\omega)}{\tau} - e\mathbf{E}(\omega).$$

Despejamos $\mathbf{p}(\omega)$:

$$\mathbf{p}(\omega) = \frac{-e\mathbf{E}(\omega)}{1/\tau - i\omega}$$

Y obtenemos:

$$\mathbf{j}(\omega) = -\frac{ne\mathbf{p}(\omega)}{m} = \frac{(ne^2/m)\mathbf{E}(\omega)}{(1/\tau) - i\omega}.$$

Introducimos la conductividad alterna: $\mathbf{j}(\omega) = \sigma(\omega)\mathbf{E}(\omega),$

Entonces:

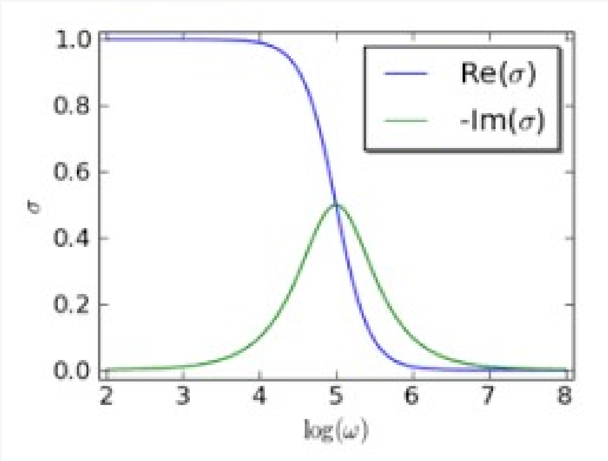
$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \quad \sigma_0 = \frac{ne^2\tau}{m}.$$

verificarlo (fácil)

Modelo de Drude

Conductividad eléctrica AC de un metal

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} = \underbrace{\frac{\sigma_0}{1+\omega^2\tau^2}}_{\text{Parte real}} + \underbrace{i\omega\tau\frac{\sigma_0}{1+\omega^2\tau^2}}_{\text{Parte imaginaria}}.$$



Complex conductivity for different frequencies assuming that $\tau = 10^{-5}$ and that $\sigma_0 = 1$.

El máximo de la parte imaginaria es en $\omega\tau = 1$

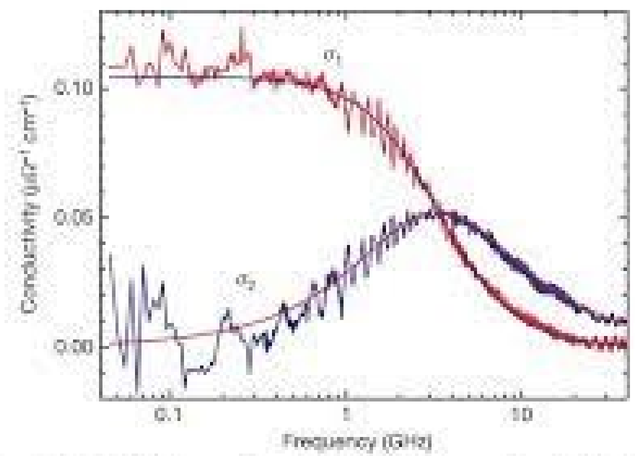


Figure 2 | Conductivity spectrum of UPd_2Al_3 at temperature 2.75 K; both real and imaginary parts (σ_1 and σ_2, respectively) are shown. The fit ($\sigma_1 + i\sigma_2 = \sigma_0(1 - i\omega\tau)^{-1}$, $\sigma_0 = 0.105\ \mu\Omega^{-1}\,cm^{-1}$, $\tau = 4.8 \times 10^{-11}$ s) documents the excellent agreement of experimental data and the Drude prediction. The characteristic relaxation rate $1/\tau$ is marked by the decrease in σ_1 and the maximum in σ_2 around 3 GHz.

Modelo de Drude

Conductividad eléctrica AC de un metal

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau} = \underbrace{\frac{\sigma_0}{1+\omega^2\tau^2}}_{\text{Parte real}} + i\omega\tau \underbrace{\frac{\sigma_0}{1+\omega^2\tau^2}}_{\text{Parte imaginaria}}.$$

$$\sigma = \frac{f_0 N e^2}{m(\gamma_0 - i\omega)} \qquad (7.58)$$

This is essentially the model of Drude (1900) for the electrical conductivity, with $f_0 N$ being the number of free electrons per unit volume in the medium. The damping constant γ_0/f_0 can be determined empirically from experimental data on the conductivity. For copper, $N \simeq 8 \times 10^{28}$ atoms/m^3 and at normal temperatures the low-frequency conductivity is $\sigma \simeq 5.9 \times 10^7\ (\Omega \cdot \text{m})^{-1}$. This gives $\gamma_0/f_0 \simeq 4 \times 10^{13}\ \text{s}^{-1}$. Assuming that $f_0 \sim 1$, this shows that up to frequencies well beyond the microwave region ($\omega \lesssim 10^{11}\ \text{s}^{-1}$) conductivities of metals are essentially real (i.e., current in phase with the field) and independent of frequency. At higher frequencies (in the infrared and beyond) the conductivity is complex and varies with frequency in a way described qualitatively by the simple result (7.58).

J.D. Jackson

Modelo de Drude

Conductividad óptica de los metales

La conductividad AC que calculamos es importante para entender las propiedades ópticas de los metales, y se la llama también *conductividad óptica*.

Supongamos que se aplica radiación electromagnética (luz) a un metal.

Actúa un campo eléctrico $\mathbf{E}(\mathbf{r}, t)$ (el campo magnético de la luz se puede ignorar en general)

Si la longitud de onda de **E**(**r**,t) es mucho mayor que el camino libre medio en el metal podemos usar la dependencia:

$$\mathbf{j}(\mathbf{r}, \omega) = \sigma(\omega)\mathbf{E}(\mathbf{r}, \omega).$$

Luz visible
- Longitud de onda: 380 -750 nanómetros (nm)
- Frecuencia: 400 -790 terahertz (Thz)

Modelo de Drude

Conductividad Óptica de los metales

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0;$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \qquad \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

$$\text{div } \mathbf{E} = 4\pi\rho$$

(Por ahora suponemos que no hay acumulación de carga)

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} = \frac{i\omega}{c}\nabla \times \mathbf{H} = \frac{i\omega}{c}\left(\frac{4\pi\sigma}{c}\mathbf{E} - \frac{i\omega}{c}\mathbf{E}\right).$$

$$-\nabla^2 \mathbf{E} = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi i\sigma}{\omega}\right)\mathbf{E}.$$

$$-\nabla^2 \mathbf{E} = \frac{\omega^2}{c^2}\epsilon(\omega)\mathbf{E},$$

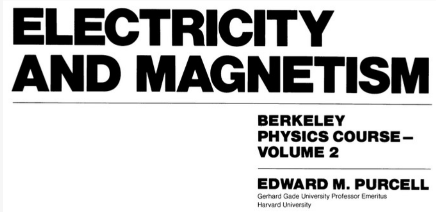
Constante/función dieléctrica

$$\epsilon(\omega) = 1 + \frac{4\pi i\sigma}{\omega}.$$

Relación entre conductividad y función dieléctrica!

Modelo de Drude

Velocidad de la luz dentro de un medio material



Substituting these into Eq. 67, we find that the equations are satisfied if

$$\frac{\omega}{k} = \frac{c}{\sqrt{\epsilon}} \qquad \text{and} \qquad B_0 = \sqrt{\epsilon} E_0 \tag{70}$$

The wave velocity ω/k differs from the velocity of light in vacuum by the factor $1/\sqrt{\epsilon}$. The electric and magnetic field amplitudes, E_0 and B_0, which are precisely equal in the wave in vacuum, here differ by the factor $\sqrt{\epsilon}$, the electric amplitude being the smaller. In other respects the wave resembles our plane wave in vacuum: $\mathbf{B}$ is perpendicular to $\mathbf{E}$, and the wave travels in the direction of $\mathbf{E} \times \mathbf{B}$. Of course, if we compare a wave in a dielectric with a wave of the same frequency in vacuum, the wavelength λ in the dielectric will be less than the vacuum wavelength by $1/\sqrt{\epsilon}$, since *frequency* $\times$ *wavelength* = *velocity*.

Light traveling through glass provides an example of the wave just described. In optics it is customary to define n, the *index of refraction* of a medium, as the ratio of the speed of light in vacuum to the speed of light in that medium. We have now discovered that n is nothing more than $\sqrt{\epsilon}$. In fact we have now laid most of the foundation for a classical theory of optics.

Modelo de Drude

Función dieléctrica: frecuencia de plasma

$$\epsilon(\omega) = 1 + \frac{4\pi i \sigma}{\omega}.$$

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \quad \sigma_0 = \frac{ne^2\tau}{m}.$$

$$\begin{aligned}
\epsilon(\omega) &= 1 + \frac{4\pi i}{\omega}\left(\frac{\sigma_0}{1 - i\omega\tau}\right) \\
\omega\tau \gg 1 \longrightarrow & \; 1 + \frac{4\pi i}{\omega}\left(\frac{\sigma_0}{-i\omega\tau}\right) \\
&= 1 - \frac{4\pi}{\omega^2\tau}\left(\frac{ne^2\tau}{m}\right) \\
&= 1 - \frac{1}{\omega^2}\left(\frac{4\pi ne^2}{m}\right) \\
&\equiv 1 - \frac{\omega_p^2}{\omega^2}
\end{aligned}$$

frecuencia alta

$$\epsilon(\omega) = 1 - \frac{\omega_p{}^2}{\omega^2} \qquad \omega_p{}^2 = \frac{4\pi ne^2}{m}$$

Frecuencia de plasma

En sistema SI de unidades: (Jackson)

$$\frac{\epsilon(\omega)}{\epsilon_0} \simeq 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega_p^2 = \frac{NZe^2}{\epsilon_0 m}$$

(NZ es la densidad)

Modelo de Drude

Análisis de la función dieléctrica

Supongamos que es válida: $$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

para frecuencia alta $\omega\tau \gg 1$

$$\omega_p^2 = \frac{4\pi n e^2}{m}$$

$$\epsilon(\omega) = 1 + \frac{4\pi i \sigma}{\omega}.$$

[1] $\omega < \omega_p$ → $\epsilon(\omega)$ es real y negativo

→ $$-\nabla^2 \mathbf{E} = \frac{\omega^2}{c^2}\epsilon(\omega)\mathbf{E}.$$

da soluciones evanescentes, no propagantes de **E**

el metal es *opaco*

[2] $\omega > \omega_p$ → $\epsilon(\omega)$ es positivo → soluciones propagantes de **E**

el metal es *transparente*

Modelo de Drude

Análisis de la función dieléctrica

$\omega < \omega_p$ → el metal es *opaco*

$\omega > \omega_p$ → el metal es *transparente*

Usando $$\tau = \left(\frac{0.22}{\rho_\mu}\right)\left(\frac{r_s}{a_0}\right)^3 \times 10^{-14}\ \text{sec.}$$

$$\omega_p\tau = 1.6 \times 10^2 \left(\frac{r_s}{a_0}\right)^{3/2}\left(\frac{1}{\rho_\mu}\right)$$

in microhm centimeters, ρ_μ, is of the order of unity or less.

r_s/a_0 is in the range from 2 to 6.

Se cumple la hipótesis de alta frecuencia alrededor de ω_p

Modelo de Drude

Análisis de la función dieléctrica

The alkali metals have, in fact, been observed to become transparent in the ultraviolet. A numerical evaluation of (1.38) gives the frequency at which transparency should set in as

$$\nu_p = \frac{\omega_p}{2\pi} = 11.4 \times \left(\frac{r_s}{a_0}\right)^{-3/2} \times 10^{15}\ \text{Hz} \qquad (1.40)$$

or

$$\lambda_p = \frac{c}{\nu_p} = 0.26 \left(\frac{r_s}{a_0}\right)^{3/2} \times 10^{3}\ \text{Å}. \qquad (1.41)$$

In Table 1.5 we list the threshold wavelengths calculated from (1.41), along with the observed thresholds.

$$\omega_p^2 = \frac{4\pi n e^2}{m} \qquad (1.38)$$

Table 1.5
OBSERVED AND THEORETICAL WAVELENGTHS BELOW WHICH THE ALKALI METALS BECOME TRANSPARENT

ELEMENT	THEORETICAL[a] λ (10^3 Å)	OBSERVED λ (10^3 Å)
Li	1.5	2.0
Na	2.0	2.1
K	2.8	3.1
Rb	3.1	3.6
Cs	3.5	4.4

[a] From Eq. (1.41).
Source: M. Born and E. Wolf, *Principles of Optics*, Pergamon, New York, 1964.

Modelo de Drude

Oscilaciones de carga: plasmones

$$\omega_p{}^2 = \frac{4\pi n e^2}{m} \quad (1.38)$$

El gas de electrones puede tener oscilaciones de la densidad de carga

$$\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t},$$

$$\nabla \cdot \mathbf{j}(\omega) = i\omega\rho(\omega).$$

$$\nabla \cdot \mathbf{E}(\omega) = 4\pi\rho(\omega).$$

$$\mathbf{j}(\mathbf{r}, \omega) = \sigma(\omega)\mathbf{E}(\mathbf{r}, \omega).$$

→

$$i\omega\rho(\omega) = 4\pi\sigma(\omega)\rho(\omega).$$

Pedimos que $\rho(\omega)$ no se anule

This has a solution provided that

$$1 + \frac{4\pi i\sigma(\omega)}{\omega} = 0,$$

O sea, pedimos que se anule : $\epsilon(\omega) = 1 + \frac{4\pi i\sigma}{\omega}$

análogo a la condición para pasar de opaco a transparente.

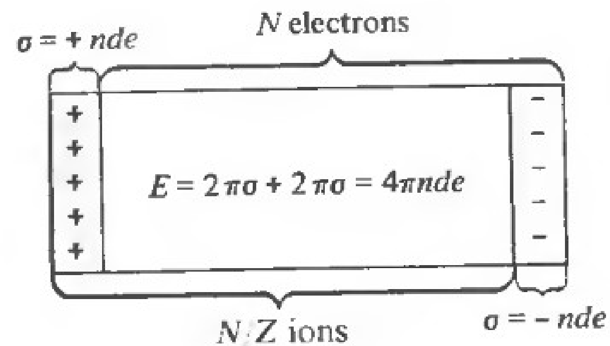
Modelo de Drude

Oscilaciones de carga: plasmones

$$\omega_p^2 = \frac{4\pi n e^2}{m}$$

The nature of this charge density wave, known as a plasma oscillation or plasmon, can be understood in terms of a very simple model.[23] Imagine displacing the entire electron gas, as a whole, through a distance d with respect to the fixed positive background of the ions (Figure 1.5).[24] The resulting surface charge gives rise to an electric field of magnitude $4\pi\sigma$, where σ is the charge per unit area[25] at either end of the slab.

Figure 1.5
Simple model of a plasma oscillation.



Consequently the electron gas as a whole will obey the equation of motion:

$$Nm\ddot{d} = -Ne|4\pi\sigma| = -Ne(4\pi nde) = -4\pi ne^2 Nd, \qquad (1.46)$$

which leads to oscillation at the plasma frequency.

Few direct observations have been made of plasmons. Perhaps the most notable is the observation of energy losses in multiples of $\hbar\omega_p$ when electrons are fired through thin, metallic films.[26] Nevertheless, the possibility of their excitation in the course of other electronic processes must always be borne in mind.

[26] C. J. Powell and J. B. Swan, *Phys. Rev.* **115**, 869 (1959).

Resumen de la Clase 3

En esta clase vimos:

Efecto Hall en el modelo de Drude.

Conductividad AC (alterna) en el modelo de Drude.

Función dieléctrica: condición de transparencia de metales

Oscilaciones y frecuencia de plasma

Guía 1: Modelo de Drude

Ejercicio 4

(a) Argumentar porqué el coeficiente de Hall definido por

revela el signo de los portadores de carga.

(b) Calcular el coeficiente de Hall en el modelo de Drude. Trabajar en el sistema de unidades SI (MKS).

$$R_H = \frac{E_y}{j_x H}.$$