Potencial de Coulomb

$$V_{(r)} = -\frac{e^2}{\epsilon r}$$

Transformada de Fourier:

$$-\frac{e^2}{\epsilon}\int_{2D} \frac{1}{r} e^{-i\bar{k}\cdot\bar{r}} d^2r$$

eligiendo el eje x en la dirección $\hat{k}$:

$$-\frac{e^2}{\epsilon}\int_0^{+\infty}\int_0^{2\pi} \frac{e^{-ikr\cos\theta}}{\cancel{r}}\,\cancel{r}\,dr\,d\theta$$

Ecuación de Bessel:

$$\boxed{\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} + \left(1-\frac{n^2}{x^2}\right)f = 0}$$

Soluciones en forma integral

$$J_n(z) = \frac{(-i)^n}{\pi} \int_0^{\pi} e^{iz\cos\theta} \cos(n\theta)\, d\theta$$

En particular

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} e^{iz\cos\theta}\, d\theta$$

tenemos

$$\int_0^{2\pi} e^{-ikr\cos\theta}\, d\theta = \int_0^{\pi} e^{-ikr\cos\theta}\, d\theta +$$

$$+ \int_{\pi}^{2\pi} e^{-ikr\cos\theta}\, d\theta$$

Propiedad: $J_n(-x) = J_{-n}(x) = (-1)^n J_n(x)$

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1 Deal, Joella Rae, "Basics of Bessel Functions" (2018). University Honors Theses. Paper 546. https://doi.org/10.15760/honors.552

Luego $J_0(-x) = J_0(x)$ entonces

$$\int_0^{2\pi} e^{-ikr\cos\theta}\, d\theta = \pi J_0(kr) + \int_\pi^{2\pi} e^{-ikr\cos\theta}\, d\theta$$

$$u = \theta - \pi$$

$$\int_\pi^{2\pi} e^{-ikr\cos\theta}\, d\theta = \int_0^{\pi} e^{-ikr\cos(u+\pi)}\, du =$$

$$\cos(u+\pi) = \cos(u)\cos(\pi) - \sin(u)\sin(\pi)$$
$$= -\cos(u)$$

$$= \int_0^{\pi} e^{ikr\cos(u)}\, du = \pi J_0(kr)$$

entonces

$$\int_0^{2\pi} e^{-ikr\cos\theta}\, d\theta = 2\pi J_0(kr)$$

De manera que nos quede

$$-\frac{e^2}{\epsilon}\int_0^{+\infty} 2\pi J_0(kr)\, dr =$$

$$x = kr \quad dx = k\,dr$$

$$-\frac{e^2 2\pi}{k\epsilon}\int_0^{+\infty} J_0(x)\, dx$$

transformada de Laplace

$$\int_0^{+\infty} f(x)\, e^{-sx}\, dx$$

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

$$\int_0^{\infty} x^{2m} e^{-sx}\, dx = \frac{(2m)!}{s^{2m+1}}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2 2^{2m}} \frac{(2m)!}{s^{2m+1}} = *$$

Pero (Taylor)

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2 2^{2m}} \frac{(2m)!}{s^{2m}} = \frac{1}{\sqrt{1+\left(\frac{1}{s}\right)^2}}$$

Entonces:

$$* = \frac{1}{s} \cdot \frac{1}{\sqrt{1+\left(\frac{1}{s}\right)^2}} = \frac{1}{\sqrt{s^2+1}}$$

como

$$\int_0^\infty J_0(x)\, e^{-sx}\, dx = \frac{1}{\sqrt{s^2+1}}$$

en el límite $s \to 0$

$$\int_0^\infty J_0(x)\, dx = 1$$

luego $\tilde{F}\left(-\frac{e^2}{\epsilon r}\right) = \frac{-2\pi e^2}{\epsilon k}$

$$V(r) = -\frac{e^2}{\epsilon r} e^{-\mu r}$$

$$\mu = \frac{1}{\lambda}$$

$$-\frac{e^2}{\epsilon} \int_{2D} \frac{e^{-\mu r} e^{-i\bar{k}\cdot\bar{r}}}{r} d^2 r =$$

$$= -\frac{e^2}{\epsilon} \int_0^\infty \int_0^{2\pi} \frac{e^{-\mu r} e^{-ikr\cos\theta}}{\cancel{r}} \cancel{r}\, dr\, d\theta$$

$$= -\frac{e^2}{\epsilon} \int_0^\infty e^{-\mu r} \int_0^{2\pi} e^{-ikr\cos\theta} d\theta\, dr$$

$$= -\frac{e^2}{\epsilon} \int_0^\infty e^{-\mu r} 2\pi J_0(kr)\, dr =$$

$$= -\frac{e^2}{\epsilon} 2\pi \int_0^\infty J_0(kr)\, e^{-\mu r}\, dr$$

$$= -\frac{e^2 2\pi}{k\epsilon} \int_0^\infty J_0(x)\, e^{-\frac{\mu}{k}x}\, dx$$

$$= -\frac{2\pi e^2}{k\epsilon} \cdot \frac{1}{\sqrt{\left(\frac{\mu}{k}\right)^2+1}} =$$

$$= -\frac{2\pi e^2}{\epsilon} \cdot \frac{1}{\sqrt{\mu^2+k^2}}$$

3D

$$V_c(k) = -\frac{4\pi e^2}{\epsilon k^2}$$

$$V_y(k) = -\frac{4\pi e^2}{\varepsilon (k^2+\mu^2)}$$