Boltzmann & transporte

Lectura: K. Huang, Cap. 5; M. Kardar Cap. 3; F. Reif. Cap 12-13.

» La ecuación de Boltzmann

$$\left(\frac{\partial}{\partial t} + \boldsymbol{F}\cdot\nabla_p + \frac{\boldsymbol{p}}{m}\cdot\nabla_r\right) f(\boldsymbol{r},\boldsymbol{p},t) = \left.\frac{\partial f}{\partial t}\right|_{\text{col}}$$
$$= \int d^3p_2 d\Omega \frac{d\sigma}{d\Omega}|v_2 - v_1| \left[f(p_1')f(p_2') - f(p_1)f(p_2)\right]$$

donde $f(\boldsymbol{r},\boldsymbol{p},t)$ es la distribución de 1 partícula, i.e, $dN = f(r,p,t)d^3r\,d^3p$ es el número de partículas con posición entre $\boldsymbol{r}$ y $\boldsymbol{r}+\boldsymbol{dr}$ y momento entre $\boldsymbol{p}$ y $\boldsymbol{p}+\boldsymbol{dp}$

» Promedios locales

Sea $\mathcal{O}(\boldsymbol{r}, \boldsymbol{p})$ vamos a trabajar con promedios sobre $\boldsymbol{p}$

$$\langle \mathcal{Q} \rangle(\mathbf{r}, t) = \frac{\int \mathcal{O}(r, p)\, f(r, p, t)\, d^3 p}{\underbrace{\int f(r, p)\, d^3 p}_{n(r,t)}}$$

$$\begin{cases} \boldsymbol{u}(r, t)\, n(r, t) = \int \frac{\boldsymbol{p}}{m} f(r, p, t)\, d^3 p \\ \epsilon(r, t)\, n(r, t) = \int \frac{m}{2} \left(\frac{\boldsymbol{p}}{m} - \boldsymbol{u}\right)^2 f(r, p, t)\, d^3 p \\ \vdots \end{cases}$$

» Leyes de Conservación

¿Qué pasa con las magnitudes (χ) que se conservan microscópicamente?

$$\chi(p_1, q, t) + \chi(p_2, q, t) = \chi(p'_1, q, t) + \chi(p'_2, q, t)$$

» Conservación del número de partículas ($\chi = 1$)

$$\int d^3p \left(\frac{\partial}{\partial t} + \boldsymbol{F}\cdot\nabla_p + \frac{\boldsymbol{p}}{m}\cdot\nabla_r\right) f(\boldsymbol{r},\boldsymbol{p},t) = \int d^3p \left.\frac{\partial f}{\partial t}\right|_{\text{col}} \rightarrow 0$$

$$\int d^3p\,\partial_t f(r,p,t) + \int d^3p\,\boldsymbol{F}\cdot\nabla_p f + \int d^3p \frac{\boldsymbol{p}}{m}\cdot\nabla_r f = 0$$

$$\partial_t n(r,t) + \nabla_r \cdot \int d^3p \frac{\boldsymbol{p}}{m} f = 0$$

$$\boxed{\partial_t n + \nabla\cdot(n\boldsymbol{u}) = 0}$$

Momento lineal $\quad \chi = \frac{\boldsymbol{p}}{m} - \boldsymbol{u}$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \frac{\boldsymbol{F}}{m} - \frac{1}{mn}\nabla\cdot\mathbb{P}$$

con

$$P_{ij}(\boldsymbol{r},t) = \int (p_i/m - u_i)(p_j - mu_j)\, f(\boldsymbol{r},\boldsymbol{p},t)\, d^3p$$

Energía cinética $\quad \chi = \frac{m}{2}(\boldsymbol{p}/m - \boldsymbol{u})^2$

La densidad local de energía cinética, $\varepsilon = \langle \frac{(p-mu)^2}{2m} \rangle$

$$\partial_t \varepsilon + \boldsymbol{u}\cdot\nabla\varepsilon = -\frac{1}{n}\nabla\cdot\boldsymbol{q} - \frac{1}{n}P_{ij}u_{ij}$$

con

$$\boldsymbol{q} = \frac{1}{m}\int (\boldsymbol{p}-m\boldsymbol{u})\frac{1}{2m}(\boldsymbol{p}-m\boldsymbol{u})^2\, f(r,p,t)\, d^3p$$

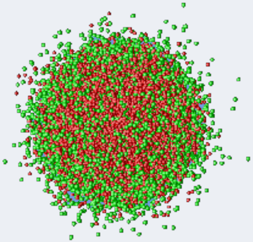
y $u_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$

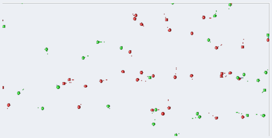
» ¿Cada cuánto choca una partícula?

El No. total de colisiones por unidad de tiempo es

$$Z = \int d^3r\, d^3p_1 d^3p_2\, \sigma\, |\boldsymbol{v}_1 - \boldsymbol{v}_2| f(r, p_1, t) f(r, p_2, t) \quad \Rightarrow \quad \tau_c \simeq \frac{N}{2Z}$$

En un sistema uniforme en equilibrio $\tau_c = \frac{1}{4n\sigma}\sqrt{\frac{m\pi}{k_B T}}$ y el camino libre medio $\lambda = \bar{v}\,\tau_c$





» Límite de colisiones muy frecuentes

Si $\tau_c \ll \tau_U$

$$f(r,p,t) = f^{\text{MB local}}[\boldsymbol{u},\beta,n] = n(r,t)\left(\frac{\beta(r,t)}{2\pi m}\right)^{3/2} e^{-\frac{\beta(r,t)}{2m}(\boldsymbol{p}-mu(r,t))^2}$$

Equilibrio **local**

$$\boldsymbol{q} = 0$$

$$P_{ij} = \delta_{ij} P = \frac{1}{3m}\int (\boldsymbol{p} - m\boldsymbol{u})^2 f(r,p,t)\, d^3p$$

$$P = \frac{1}{3m}\int p^2 n \left(\frac{\beta}{2\pi m}\right)^{3/2} e^{-\frac{\beta}{2m}p^2}\, d^3p$$

$$= \frac{1}{\cancel{3}m}\cancel{3}\frac{mn}{\beta} = n(r,t) k_B T(r,t)$$

$$\varepsilon = \frac{3}{2}\, n(\boldsymbol{r},t)\, k_B T(\boldsymbol{r},t)$$

$$\partial_t n + \nabla\cdot(n\boldsymbol{u}) = 0, \qquad m\left[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = \boldsymbol{F} - \frac{\nabla P}{n}, \qquad \left[\partial_t T + \boldsymbol{u}\cdot\nabla T\right] = -\frac{2}{3}T\nabla\cdot\boldsymbol{u}$$

¿Qué pasa con la ecuación de $T(r,t)$?

$$\partial_t n + (\boldsymbol{u}\cdot\boldsymbol{\nabla})n = -n\boldsymbol{\nabla}\cdot\boldsymbol{u} \quad \text{y} \quad -\frac{3}{2}\frac{n}{T}\left[\partial_t T + (\boldsymbol{u}\cdot\boldsymbol{\nabla})T\right] = n\boldsymbol{\nabla}\cdot\boldsymbol{u}$$

$$\partial_t n - \frac{3}{2}\frac{n}{T}\partial_t T + (\boldsymbol{u}\cdot\boldsymbol{\nabla})n - \frac{3}{2}\frac{n}{T}(\boldsymbol{u}\cdot\boldsymbol{\nabla})T = 0$$

$$D[n] - \frac{3}{2}\frac{n}{T}D[T] = 0 \quad \text{con} \quad D[n] = \partial_t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})n$$

$$0 = \frac{1}{T^{3/2}}\left[D[n] - \frac{3}{2}\frac{n}{T}D[T]\right]$$

$$\frac{n}{T^{3/2}} = \text{constante} \qquad S = -k_B H = N\left\{\ln\left[n\left(\frac{\beta}{2\pi m}\right)^{3/2}\right] - \frac{3}{2}\right\}$$

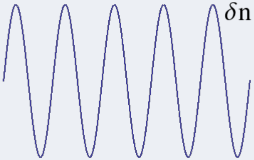
Propagación de Ondas

Si $\boldsymbol{u} = O(1), n(\boldsymbol{r}, \boldsymbol{t}) = n_0(\boldsymbol{r}) + \delta n(\boldsymbol{r}, \boldsymbol{t})$

$$\partial_t + \boldsymbol{\nabla} \cdot (n\boldsymbol{u}) = 0 \quad \rightarrow \quad \partial_t \delta n + \boldsymbol{\nabla} \cdot (n_0 \boldsymbol{u}) = 0$$

$$m\left[\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right] = -\frac{\nabla P}{n} = -\frac{\nabla(n k_B T)}{n} \quad \rightarrow$$

$$m n_0 \partial_t \boldsymbol{u} + \frac{\partial P}{\partial n}\Big|_S \nabla \delta n = 0$$



Con lo cual

$$\partial_t^2 \delta n - \frac{1}{m} \nabla \cdot \left(n_0 \frac{\partial P}{\partial n}\bigg|_s (n_0) \nabla \delta n \right) = 0 \qquad \text{si } n_0 = \text{cte}$$

$$\boxed{\frac{\partial^2}{\partial t^2} \delta n = c_s^2 \nabla^2 \delta n}$$

con $c_s = n_0 \frac{\partial P}{\partial n}\big|_s (n_0)$ es la velocidad del sonido.

» Cerca del hidrodinámico – Aproximación del tiempo de relajación

Si $f(r,p,t) = \overbrace{f^{(0)}(r,p,t)}^{\text{local}} + g(r,p,t)$. Así

$$\left.\frac{\partial f}{\partial t}\right|_{\text{col}} = \int d^3p_2 d\Omega \frac{d\sigma}{d\Omega}|\boldsymbol{v}_2 - \boldsymbol{v}_1| \left[f(p_1')f(p_2') - f(p_1)f(p_2)\right]$$

$$\simeq \int d^3p_2 d\Omega \frac{d\sigma}{d\Omega}|\boldsymbol{v}_2 - \boldsymbol{v}_1| \Big[f^{(0)}(p_1')g(p_2') + g(p_1')f^{(0)}(p_2')$$

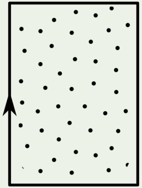
$$-f^{(0)}(p_1)g(p_2) - g(p_1)f^{(0)}(p_2)\Big]$$

En la aproximación del tiempo de relajación (τ) tenemos

$$\left.\frac{\partial f}{\partial t}\right|_{\text{col}} \simeq -\frac{g}{\tau} = -\frac{f - f^{(0)}}{\tau}$$

$$\left(\partial_t + \boldsymbol{F}\cdot\nabla_p + \frac{\boldsymbol{p}}{m}\cdot\nabla_r\right)(f^{(0)} + g) \simeq -\frac{g}{\tau}$$

Flujo de calor – Conductividad térmica



$$\boldsymbol{q} = \int \frac{\boldsymbol{p}}{m}\frac{p^2}{2m} f 3p = \int \frac{\boldsymbol{p}}{m}\frac{p^2}{2m} g 3p$$

$$= -\tau \int \frac{\boldsymbol{p}}{m}\frac{p^2}{2m}\left(\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{\nabla}_r f^{(0)}\right) d^3p$$

$$\boldsymbol{q} = -\frac{5}{2}\frac{\tau}{m}\nabla\left(\frac{n}{\beta^2}\right) = -\kappa\nabla T$$ usando que $\nabla(P) = \nabla(nK_BT) = 0$ (**$\boldsymbol{u}$=0**)

donde κ define la conductividad térmica.